18.152 PROBLEM SET 1

due February 14th 9:30 am

or please submit to Gradescope by 16th Saturday 1:00 pm

You can collaborate with other students when working on problems. However, you should write the solutions using your own words and thought.

Problem 1. *Let $g(x), h_1(t), h_2(t), f(x,t)$ be smooth functions and let $\alpha \geq 0$ be a non-negative constant. Then, the following Cauchy-Robin problem to the diffusion equation has at most one smooth solution.*

$$
\begin{aligned}
&u_t(x,t) = u_{xx}(x,t) + f(x,t), \qquad \textit{for } 0 \leq x \leq L, 0 \leq t,\\
&- u_x(0,t) + \alpha\, u(0,t) = h_1(t), \qquad \textit{for } 0 \leq t,\\
&u_x(L,t) + \alpha\, u(L,t) = h_2(t), \qquad \textit{for } 0 \leq t,\\
&u(x,0) = g(x), \qquad \textit{for } 0 \leq x \leq L.
\end{aligned}
$$

Notice that if $\alpha = 0$ then it is a Cauchy-Neumann problem.

Problem 2. *Given smooth $g(x)$, find the all solutions to the following Cauchy-Neumann problem;*

$$
\begin{aligned}
&u_t(x,t) = u_{xx}(x,t), \qquad \textit{for } 0 \leq x \leq L, 0 \leq t,\\
&- u_x(0,t) = -1, \qquad \textit{for } 0 \leq t,\\
&u_x(L,t) = 2, \qquad \textit{for } 0 \leq t,\\
&u(x,0) = g(x), \qquad \textit{for } 0 \leq x \leq L.
\end{aligned}
$$

Hint: Refer the uniqueness result in the previous result, and consider a function $\tilde{u}(x,t) = u(x,t) + ax^2 + bx + ct$ satisfying $\tilde{u}_t = \tilde{u}_{xx}$ and $\tilde{u}_x(0,t) = \tilde{u}_x(L,t) = 0$.

Problem 3. *Suppose that $u(x,t)$ is the smooth solution to the following Cauchy-Neumann problem;*

$$
\begin{aligned}
&u_t(x,t) = u_{xx}(x,t), \qquad \textit{for } 0 \leq x \leq L, 0 \leq t,\\
&u_x(0,t) = u_x(L,t) = 0, \qquad \textit{for } 0 \leq t,\\
&u(x,0) = g(x), \qquad \textit{for } 0 \leq x \leq L,
\end{aligned}
$$

where $g(x)$ is smooth. Then, for $p \geq 2$ prove the following inequality

$$\frac{d}{dt}\int_0^L |u(x,t)|^p dx \leq 0,$$

namely $\sqrt[p]{\int_0^L |u(x,t)|^p dx}$ monotonically decreases.

Notice that a smooth function $u(x,t)$ satisfies

$$(*) \qquad \lim_{p\to\infty}\left(\int_0^L |u(x,t)|^p dx\right)^{\frac{1}{p}} = \sup_{0\leq x\leq L}|u(x,t)|.$$

Hence, the result in Problem 3 shows that $\sup_{0\le x\le L}|u(x,t)|$ monotonically decreases. We will prove it again in class by using another method without (*).

Problem 4. *Let $u(x,t)$ be the smooth solution to the following Cauchy-Neumann problem;*

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \qquad \textit{for } 0 \le x \le L, 0 \le t,\\ &u_x(0,t) = u_x(L,t) = 0, \qquad \textit{for } 0 \le t,\\ &u(x,0) = g(x), \qquad \textit{for } 0 \le x \le L, \end{aligned}$$

where $g(x)$ is smooth. Show the following inequality

$$\frac{d}{dt}\int_0^L |u_x(x,t)|^2 + |u_t(x,t)|^2 dx \le 0.$$